

CHARACTERIZATION OF PARABOLIC HARDY SPACES BY LITTLEWOOD-PALEY FUNCTIONS

SHUICHI SATO

ABSTRACT. We consider Littlewood-Paley functions associated with non-isotropic dilations. We prove that they can be used to characterize the parabolic Hardy spaces of Calderón-Torchinsky.

1. INTRODUCTION

Let P be an $n \times n$ real matrix such that

$$\langle Px, x \rangle \geq \langle x, x \rangle \quad \text{for all } x \in \mathbb{R}^n,$$

where $\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n$ is the inner product in \mathbb{R}^n . Let $\gamma = \text{trace } P$. Define a dilation group $\{A_t\}_{t>0}$ on \mathbb{R}^n by $A_t = t^P = \exp((\log t)P)$. It is known that $|A_t x|$ is strictly increasing as a function of t on $\mathbb{R}_+ = (0, \infty)$ for $x \neq 0$, where $|x| = \langle x, x \rangle^{1/2}$. Define a norm function $\rho(x)$ to be the unique positive real number t such that $|A_{t^{-1}} x| = 1$ when $x \neq 0$ and $\rho(0) = 0$. Then $\rho(A_t x) = t\rho(x)$, $t > 0$, $x \in \mathbb{R}^n$, $\rho \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and the following results are known (see [4, 6]):

- (P.1) $\rho(x + y) \leq \rho(x) + \rho(y)$;
- (P.2) $\rho(x) \leq 1$ if and only if $|x| \leq 1$;
- (P.3) $|x| \leq \rho(x)$ if $|x| \leq 1$;
- (P.4) $|x| \geq \rho(x)$ if $|x| \geq 1$;
- (P.5) $|A_t x| \geq t|x|$ if $t \geq 1$ for all $x \in \mathbb{R}^n$;
- (P.6) $|A_t x| \leq t|x|$ if $0 < t \leq 1$ for all $x \in \mathbb{R}^n$.

Similarly, we can consider a norm function $\rho^*(x)$ associated with the dilation group $\{A_t^*\}_{t>0}$, where A_t^* denotes the adjoint of A_t . We have properties analogous to those for $\rho(x)$, A_t above.

Let

$$(1.1) \quad g_\varphi(f)(x) = \left(\int_0^\infty |f * \varphi_t(x)|^2 \frac{dt}{t} \right)^{1/2}$$

be the Littlewood-Paley function on \mathbb{R}^n , where $\varphi_t(x) = t^{-\gamma} \varphi(A_t^{-1} x)$ and φ is a function in $L^1(\mathbb{R}^n)$ such that

$$(1.2) \quad \int_{\mathbb{R}^n} \varphi(x) dx = 0.$$

For L^p boundedness of g_φ , $1 < p < \infty$, see [1, 15].

2010 *Mathematics Subject Classification*. Primary 42B25; Secondary 42B30.

Key Words and Phrases. Littlewood-Paley functions, parabolic Hardy spaces.

The author is partly supported by Grant-in-Aid for Scientific Research (C) No. 25400130, Japan Society for the Promotion of Science.

We say that a tempered distribution f belongs to the parabolic H^p if $\|f\|_{H^p} = \|f^*\|_p < \infty$, where $f^*(x) = \sup_{t>0} |\Phi_t * f(x)|$ and $\|f^*\|_p = \|f^*\|_{L^p}$, with $\Phi \in \mathcal{S}(\mathbb{R}^n)$ satisfying $\int \Phi(x) dx = 1$, $\text{supp}(\Phi) \subset \{|x| \leq 1\}$ (see [4, 5], [8]). We have denoted by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz class of rapidly decreasing smooth functions on \mathbb{R}^n .

Let $\varphi \in L^1(\mathbb{R}^n)$. We consider the non-degeneracy condition:

$$(1.3) \quad \sup_{t>0} |\hat{\varphi}(A_t^* \xi)| > 0 \quad \text{for all } \xi \neq 0,$$

where the Fourier transform is defined as

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx.$$

In this note we shall prove the following.

Theorem 1.1. *Let φ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying (1.2) and (1.3). Let $0 < p \leq 1$. Then if $f \in H^p$, we have*

$$c_1 \|f\|_{H^p} \leq \|g_\varphi(f)\|_p \leq c_2 \|f\|_{H^p}$$

with some positive constants c_1, c_2 independent of f .

We recall some related results when $P = E$ (the identity matrix) and g_φ is defined by $\varphi_t(x) = t^{-n} \varphi(t^{-1}x)$ in (1.1). Then Theorem 1.1 is known (see [19] and also [11] for some background materials). Let $Q(x) = [(\partial/\partial t)P(x, t)]_{t=1}$, where

$$P(x, t) = c_n \frac{t}{(|x|^2 + t^2)^{(n+1)/2}}, \quad c_n = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}},$$

is the Poisson kernel associated with the upper half space $\mathbb{R}^n \times (0, \infty)$ (see [17, Chap. I]). We note that $\hat{Q}(\xi) = -2\pi|\xi|e^{-2\pi|\xi|}$. Then it is also known that

$$(1.4) \quad c_1 \|f\|_{H^p} \leq \|g_Q(f)\|_p \leq c_2 \|f\|_{H^p}$$

for $f \in H^p(\mathbb{R}^n)$, $0 < p < \infty$, with positive constants c_1, c_2 (see [8] and also [19]). In [9, Chap. 7] we can find a relation between Hardy spaces and Littlewood-Paley functions associated with the heat kernel on homogeneous groups.

Uchiyama [19] gave a proof of the first inequality of (1.4) for $0 < p \leq 1$ by methods of real analysis without the use of special properties of the Poisson kernel such as harmonicity, a semigroup property. Applying a similar argument, [19] proved the first inequality of the conclusion of Theorem 1.1 (when $P = E$) for $0 < p \leq 1$:

$$(1.5) \quad \|f\|_{H^p} \leq c \|g_\varphi(f)\|_p.$$

For a function F on \mathbb{R}^n and positive real numbers N, R , define the Peetre maximal function $F_{N,R}^{**}$ by

$$F_{N,R}^{**}(x) = \sup_{y \in \mathbb{R}^n} \frac{|F(x-y)|}{(1+R|y|)^N}$$

(see [13]). Then, it is known that the maximal function $F_{N,R}^{**}$ can be used along with well-known arguments to prove (1.5) for $0 < p \leq 1$ when $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies a non-degeneracy condition and the condition $\text{supp}(\hat{\varphi}) \subset \{a_1 \leq |\xi| \leq a_2\}$ for $a_1, a_2 > 0$.

In [16], (1.5) was proved for f in a dense subspace of $H^p(\mathbb{R}^n)$ and for φ in a class of functions including Q and a general $\varphi \in \mathcal{S}(\mathbb{R}^n)$, without the restriction on $\text{supp}(\hat{\varphi})$ above, with (1.2) and (1.3) by applying a vector valued inequality related to the Littlewood-Paley theory. The proof of the vector valued inequality is based

on an application of the maximal function $F_{N,R}^{**}$. This proof of (1.5) was discovered by [16].

The purpose of this note is to generalize the methods of [16] to the case of the parabolic Hardy spaces and get the characterization of the parabolic Hardy spaces in terms of Littlewood-Paley functions (Theorem 1.1).

In Section 2, we shall prove an analogue of the vector valued inequality in [16] for the general dilation group $\{A_t\}$ (Theorem 2.4). We shall consider g_φ for φ in a class of functions which includes those φ considered in Theorem 1.1 and prove (1.5) in the case of the parabolic H^p for such φ and for f in a dense subspace of H^p as an application of Theorem 2.4 (Corollary 3.1). Theorem 2.4 will be stated more generally than needed for the proof of Corollary 3.1 as weighted vector valued inequalities.

In Section 3, a proof of Corollary 3.1 will be given by applying Theorem 2.4 and an atomic decomposition for Hardy spaces. Also, Theorem 1.1 will be derived from Corollary 3.1.

2. WEIGHTED VECTOR VALUED INEQUALITIES WITH NON-ISOTROPIC DILATIONS

We need a partition of unity on $\mathbb{R}^n \setminus \{0\}$ associated with φ satisfying (1.3).

Lemma 2.1. *Suppose that φ is a function in $L^1(\mathbb{R}^n)$ satisfying (1.3). We assume that $\hat{\varphi} \in C^\infty(\mathbb{R}^n \setminus \{0\})$. Then, there exist $b_0 \in (0, 1)$ and $r_1, r_2 > 0$, $r_1 < r_2$, such that for any $b \in [b_0, 1)$ there exists a function η with the following properties:*

- (1) $\eta \in C^\infty(\mathbb{R}^n)$;
- (2) $\hat{\eta} \in C^\infty(\mathbb{R}^n)$ and $\text{supp } \hat{\eta} \subset \{r_1 < \rho^*(\xi) < r_2\}$;
- (3) $\sum_{j=-\infty}^{\infty} \hat{\varphi}(A_{b^j}^* \xi) \hat{\eta}(A_{b^j}^* \xi) = 1$ for $\xi \in \mathbb{R}^n \setminus \{0\}$.

When P is the identity matrix, this is in [18, Chap. V]; see also [4].

Proof of Lemma 2.1. Let $S^{n-1} = \{\xi : |\xi| = 1\}$. Since $\hat{\varphi}$ is continuous, by a compactness argument we can find a finite family $\{I_h\}_{h=1}^L$ of compact intervals in $(0, \infty)$ such that

$$\inf_{\xi \in S^{n-1}} \max_{1 \leq h \leq L} \inf_{t \in I_h} |\hat{\varphi}(A_t^* \xi)|^2 \geq c$$

with a positive constant c .

We observe that there exists $b_0 \in (0, 1)$ such that if $b \in [b_0, 1)$, $t > 0$ and $1 \leq h \leq L$, then we have $b^j t \in I_h$ for some $j \in \mathbb{Z}$ (the set of integers). This can be seen by taking $b_0 = \max_{1 \leq h \leq L} (a_h/b_h)$, where $I_h = [a_h, b_h]$.

Consider an interval $[m, H]$ in $(0, \infty)$ such that $\cup_{h=1}^L I_h \subset [m, H]$ and choose $\theta \in C_0^\infty(\mathbb{R})$ such that $\theta = 1$ on $[m, H]$, $\text{supp } \theta \subset [m/2, 2H]$, $\theta \geq 0$. Define

$$\Psi(\xi) = \sum_{j=-\infty}^{\infty} \theta(b^j \rho^*(\xi)) |\hat{\varphi}(A_{b^j}^* \xi)|^2.$$

Then $\Psi(\xi) \geq c > 0$ for $\xi \neq 0$. Note that $\Psi(A_{b^k}^* \xi) = \Psi(\xi)$ for $k \in \mathbb{Z}$. Let

$$\hat{\eta}(\xi) = \theta(\rho^*(\xi)) \overline{\hat{\varphi}(\xi)} \Psi(\xi)^{-1} \quad \text{for } \xi \neq 0$$

and $\hat{\eta}(0) = 0$, where $\overline{\hat{\varphi}(\xi)}$ denotes the complex conjugate. Then, η satisfies all the properties stated in the lemma. This completes the proof. \square

To state our results, we introduce a class of functions.

Definition 2.2. Let φ be as in Lemma 2.1. Then, we say $\varphi \in B$ if the following conditions are satisfied:

$$(2.1) \quad \varphi \in C^1(\mathbb{R}^n), \quad \partial_k \varphi \in L^1(\mathbb{R}^n), \quad 1 \leq k \leq n;$$

$$(2.2) \quad |\hat{\varphi}(\xi)| \leq C|\xi|^\epsilon \quad \text{for some } \epsilon > 0;$$

$$(2.3) \quad |\partial_\xi^\alpha \hat{\varphi}(\xi)| \leq C_{\alpha, \tau} |\xi|^{-\tau} \quad \text{outside a neighborhood of the origin,}$$

for all α and $\tau > 0$ with a constant $C_{\alpha, \tau}$, where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, $\alpha_j \in \mathbb{Z}$, $\alpha_j \geq 0$, $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $\partial_\xi^\alpha = \partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_n}^{\alpha_n}$, $\partial_{\xi_k} = \partial/\partial_{\xi_k}$.

If $\psi \in \mathcal{S}(\mathbb{R}^n)$ with (1.2) and (1.3), then clearly $\psi \in B$.

Let $\varphi \in B$. Then (2.1) implies that

$$(2.4) \quad \mathcal{F}(\partial_k \varphi)(\xi) = \Xi_k(\xi) \hat{\varphi}(\xi), \quad 1 \leq k \leq n,$$

where $\Xi_k(\xi) = 2\pi i \xi_k$ and $\partial_k = \partial_{x_k}$. Let $b \in [b_0, 1)$ and η be as in Lemma 2.1. Define

$$\hat{\zeta}(\xi) = 1 - \sum_{j \geq 0} \hat{\varphi}(A_{b^j}^* \xi) \hat{\eta}(A_{b^j}^* \xi).$$

Then $\text{supp}(\hat{\zeta}) \subset \{\rho^*(\xi) \leq r_2\}$, $\hat{\zeta} = 1$ in $\{\rho^*(\xi) < r_1\}$ and by (2.4) we have

$$\begin{aligned} \mathcal{F}(\partial_k \varphi)(\xi) &= \sum_{j \geq 0} \mathcal{F}(\partial_k \varphi)(\xi) \hat{\varphi}(A_{b^j}^* \xi) \hat{\eta}(A_{b^j}^* \xi) + \hat{\zeta}(\xi) \mathcal{F}(\partial_k \varphi)(\xi) \\ &= \sum_{j \geq 0} \hat{\varphi}(A_{b^j}^* \xi) \mathcal{F}(\alpha_{(k)}^{(b^j)})(A_{b^j}^* \xi) + \hat{\varphi}(\xi) \mathcal{F}(\beta_{(k)})(\xi), \end{aligned}$$

where $\alpha_{(k)}^{(b^j)}(x) = (\partial_k \varphi)_{b^{-j}} * \eta(x)$ and $\mathcal{F}(\beta_{(k)})(\xi) = \hat{\zeta}(\xi) \Xi_k(\xi)$.

Thus we have

$$(2.5) \quad |F(\partial_k \varphi, f)(x, t)| \leq \sum_{j \geq 0} |F(\alpha_{(k)}^{(b^j)} * \varphi, f)(x, b^j t)| + |F(\beta_{(k)} * \varphi, f)(x, t)|,$$

where $f \in \mathcal{S}(\mathbb{R}^n)$ and $F(\psi, f)(x, t) = f * \psi_t(x)$. We also write $F(\psi, f)(x, t) = F_\psi(x, t)$ when f is fixed.

Let

$$(2.6) \quad C_0(\partial_k \varphi, t, L, x) = (1 + \rho(x))^L \left| \int \mathcal{F}(\partial_k \varphi)(A_{t^{-1}}^* \xi) \hat{\eta}(\xi) e^{2\pi i \langle x, \xi \rangle} d\xi \right|,$$

with $t > 0, L \geq 0$. Then

$$|\alpha_{(k)s}^{(b^j)}(x)| = C_0(\partial_k \varphi, b^j, L, A_s^{-1} x) s^{-\gamma} (1 + \rho(x)/s)^{-L}$$

for $j \in \mathbb{Z}$. Similarly,

$$|\beta_{(k)s}(x)| = D(\Xi_k, L, A_s^{-1} x) s^{-\gamma} (1 + \rho(x)/s)^{-L},$$

with

$$(2.7) \quad D(\Xi_k, L, x) = (1 + \rho(x))^L \left| \int \hat{\zeta}(\xi) \Xi_k(\xi) e^{2\pi i \langle x, \xi \rangle} d\xi \right|.$$

Put $C(\partial_k \varphi, j, L, x) = C_0(\partial_k \varphi, b^j, L, x)$, $j \in \mathbb{Z}$. Define

$$(2.8) \quad C(\partial_k \varphi, j, L) = \int_{\mathbb{R}^n} C(\partial_k \varphi, j, L, x) dx, \quad j \in \mathbb{Z},$$

$$(2.9) \quad D(\Xi_k, L) = \int_{\mathbb{R}^n} D(\Xi_k, L, x) dx.$$

We can consider $C(\psi, j, L)$ for other $\psi \in L^1(\mathbb{R}^n)$ by (2.6) and (2.8) with ψ in place of $\partial_k \varphi$. We also write $C(\psi, j, L) = C_\varphi(\psi, j, L)$, $D(\Xi_k, L) = D_\varphi(\Xi_k, L)$ to indicate that these quantities are based on φ . We have $D(\Xi_k, L), C(\partial_k \varphi, j, L) < \infty$ for all j, L , which can be seen from Lemma 2.9 below.

Let

$$(2.10) \quad F_{N,R}^{**}(x) = \sup_{y \in \mathbb{R}^n} \frac{|F(x-y)|}{(1+R\rho(y))^N}$$

for a function F on \mathbb{R}^n and positive real numbers N, R . We need the following result in proving Theorem 2.4 below.

Lemma 2.3. *Let $\varphi \in B$, $b \in [b_0, 1)$, $N > 0$. Then we have*

$$\begin{aligned} F(\partial_k \varphi, f)(\cdot, t)_{N,t-1}^{**}(x) &\leq C \sum_{j \geq 0} C(\partial_k \varphi, j, N) b^{-jN} F(\varphi, f)(\cdot, b^j t)_{N,(b^j t)-1}^{**}(x) \\ &\quad + CD(\Xi_k, N) F(\varphi, f)(\cdot, t)_{N,t-1}^{**}(x) \end{aligned}$$

for $1 \leq k \leq n$, where $f \in \mathcal{S}(\mathbb{R}^n)$.

Proof. By (2.5) we have

$$\begin{aligned} |F_{\partial_k \varphi}(z, t)| &\leq C \sum_{j \geq 0} \int |F_\varphi(y, b^j t)| \left(1 + \frac{\rho(z-y)}{b^j t}\right)^{-N} C(\partial_k \varphi, j, N, A_{b^j t}^{-1}(z-y)) (b^j t)^{-\gamma} dy \\ &\quad + C \int |F_\varphi(y, t)| \left(1 + \frac{\rho(z-y)}{t}\right)^{-N} D(\Xi_k, N, A_t^{-1}(z-y)) t^{-\gamma} dy. \end{aligned}$$

Multiplying both sides of the inequality by $(1 + \rho(x-z)/t)^{-N}$ and noting that

$$\left(1 + \frac{\rho(z-y)}{b^j t}\right)^{-N} \left(1 + \frac{\rho(x-z)}{t}\right)^{-N} \leq C_{A,N} b^{-Nj} \left(1 + \frac{\rho(x-y)}{b^j t}\right)^{-N}$$

for any $x, y, z \in \mathbb{R}^n$ and $t > 0$ when $b^j \leq A$, we have

$$\begin{aligned}
& |F_{\partial_k \varphi}(z, t)|(1 + \rho(x - z)/t)^{-N} \\
& \leq C \sum_{j \geq 0} b^{-Nj} \int |F_\varphi(y, b^j t)| \left(1 + \frac{\rho(x - y)}{b^j t}\right)^{-N} C(\partial_k \varphi, j, N, A_{b^j t}^{-1}(z - y))(b^j t)^{-\gamma} dy \\
& \quad + C \int |F_\varphi(y, t)| \left(1 + \frac{\rho(x - y)}{t}\right)^{-N} D(\Xi_k, N, A_t^{-1}(z - y))t^{-\gamma} dy \\
& \leq C \sum_{j \geq 0} b^{-Nj} F_\varphi(\cdot, b^j t)_{N, (b^j t)^{-1}}^{**}(x) \int C(\partial_k \varphi, j, N, A_{b^j t}^{-1}(z - y))(b^j t)^{-\gamma} dy \\
& \quad + C F_\varphi(\cdot, t)_{N, t^{-1}}^{**}(x) \int D(\Xi_k, N, A_t^{-1}(z - y))t^{-\gamma} dy \\
& \leq C \sum_{j \geq 0} C(\partial_k \varphi, j, N) b^{-Nj} F_\varphi(\cdot, b^j t)_{N, (b^j t)^{-1}}^{**}(x) + C D(\Xi_k, N) F_\varphi(\cdot, t)_{N, t^{-1}}^{**}(x).
\end{aligned}$$

Taking the supremum in z over \mathbb{R}^n , we reach the conclusion. \square

Let $\varphi \in B$. Then we have

$$(2.11) \quad \sup_{j \geq 0} C_\varphi(\nabla \varphi, j, L) b^{-\tau j} < \infty,$$

for all $L, \tau > 0$, where we write $\nabla \varphi = (\partial_1 \varphi, \dots, \partial_n \varphi)$, $C_\varphi(\nabla \varphi, j, L) = \sum_{k=1}^n C_\varphi(\partial_k \varphi, j, L)$ and

$$(2.12) \quad D_\varphi(L) < \infty$$

for all $L > 0$, where $D_\varphi(L) = \sum_{k=1}^n D_\varphi(\Xi_k, L)$. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$. Then, we also note that

$$(2.13) \quad \sup_{j: b^j \leq r_2} C_\varphi(\psi, j, L) b^{-\tau j} < \infty \quad \text{for any } L, \tau > 0.$$

These results are in Lemma 2.9 below, which will be used in what follows.

We consider a ball in \mathbb{R}^n with center x and radius t relative to ρ defined by

$$B(x, t) = \{y \in \mathbb{R}^n : \rho(x - y) < t\}.$$

We say that a weight function w belongs to the class A_p , $1 < p < \infty$, of Muckenhoupt if

$$[w]_{A_p} = \sup_B \left(|B|^{-1} \int_B w(x) dx \right) \left(|B|^{-1} \int_B w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty,$$

where the supremum is taken over all balls B in \mathbb{R}^n and $|B|$ denotes the Lebesgue measure of B . Let $A_\infty = \cup_{p > 1} A_p$. Also, we define the class A_1 to be the family of weight functions w such that $M(w) \leq Cw$ almost everywhere. We denote by $[w]_{A_1}$ the infimum of all such C . Here, M is the Hardy-Littlewood maximal operator relative to ρ

$$M(f)(x) = \sup_{x \in B} |B|^{-1} \int_B |f(y)| dy,$$

where the supremum is taken over all balls B in \mathbb{R}^n containing x . (See [2, 10].)

For a weight w , we denote by $\|f\|_{p,w}$ the weighted L^p norm

$$\left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p}.$$

Then we have the following result.

Theorem 2.4. *Let $\varphi \in B$. Suppose that $0 < p, q < \infty$ and $w \in A_\infty$. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$. Suppose that $\hat{\psi} = 0$ in a neighborhood of the origin. Then*

$$\left\| \left(\int_0^\infty |f * \psi_t|^q \frac{dt}{t} \right)^{1/q} \right\|_{p,w} \leq C \left\| \left(\int_0^\infty |f * \varphi_t|^q \frac{dt}{t} \right)^{1/q} \right\|_{p,w}$$

for $f \in \mathcal{S}(\mathbb{R}^n)$ with a positive constant C independent of f .

To prove Theorem 2.4 we first show the following.

Lemma 2.5. *Let $0 < q < \infty$, $N > 0$. Suppose that $\varphi \in B$, $f \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$\int_0^\infty F(\varphi, f)(\cdot, t)_{N, t^{-1}}^{**}(x)^q \frac{dt}{t} \leq C \int_0^\infty M(|f * \varphi_t|^r)(x)^{q/r} \frac{dt}{t}, \quad r = \gamma/N.$$

We apply the next result to show Lemma 2.5.

Lemma 2.6. *Let $N = \gamma/r$, $r > 0$ and let $\varphi \in L^1(\mathbb{R}^n)$, $f \in \mathcal{S}(\mathbb{R}^n)$. Suppose that φ satisfies (2.1). Then*

$$(f * \varphi_t)_{N, t^{-1}}^{**}(x) \leq C\delta^{-N} M(|f * \varphi_t|^r)(x)^{1/r} + C\delta |f * (\nabla \varphi)_t|_{N, t^{-1}}^{**}(x)$$

for all $\delta \in (0, 1]$ with a constant C independent of δ and $t > 0$, where $f * (\nabla \varphi)_t = (f * (\partial_1 \varphi)_t, \dots, f * (\partial_n \varphi)_t)$.

To prove Lemma 2.6, we use the following.

Lemma 2.7. *Let $F \in C^1(\mathbb{R}^n)$ and $R > 0$, $N = \gamma/r$, $r > 0$. Then*

$$F_{N,1}^{**}(x) \leq C\delta^{-N} M(|F|^r)(x)^{1/r} + C\delta |\nabla F|_{N,1}^{**}(x)$$

for all $\delta \in (0, 1]$ with a constant C independent of δ , where $F_{N,R}^{**}$ is as in (2.10).

Proof. Let $\int_{B(x,t)} f(y) dy = |B(x,t)|^{-1} \int_{B(x,t)} f(y) dy$. Then, for $\delta \in (0, 1]$, $r > 0$ and $x, z \in \mathbb{R}^n$ we write

$$|F(x-z)| = \left(\int_{B(x-z,\delta)} |F(y) + (F(x-z) - F(y))|^r dy \right)^{1/r}.$$

This is bounded by

$$C_r \left(\int_{B(x-z,\delta)} |F(y)|^r dy \right)^{1/r} + C_r \left(\int_{B(x-z,\delta)} |F(x-z) - F(y)|^r dy \right)^{1/r},$$

where $C_r = 1$ if $r \geq 1$ and $C_r = 2^{-1+1/r}$ if $0 < r < 1$. Thus we have

$$|F(x-z)| \leq C_r \left(\int_{B(x-z,\delta)} |F(y)|^r dy \right)^{1/r} + C_r \sup_{y: \rho(x-z-y) < \delta} |x-z-y| |\nabla F(y)|.$$

If $|x-z-y| \leq 1$, $|x-z-y| \leq \rho(x-z-y)$ by (P.3). So, we see that

$$(2.14) \quad |F(x-z)| \leq C_r \left(\int_{B(x-z,\delta)} |F(y)|^r dy \right)^{1/r} + C_r \sup_{y: \rho(x-z-y) < \delta} \delta |\nabla F(y)|.$$

If $\rho(x - z - y) < \delta$, $\rho(x - y) < \delta + \rho(z)$. Therefore

$$\begin{aligned} |\nabla F(y)| &\leq \frac{|\nabla F(x + (y - x))|}{(1 + \rho(x - y))^N} (1 + \delta + \rho(z))^N \\ &\leq |\nabla F|_{N,1}^{**}(x) (1 + \delta + \rho(z))^N \\ &\leq 2^N |\nabla F|_{N,1}^{**}(x) (1 + \rho(z))^N. \end{aligned}$$

Thus

$$(2.15) \quad \sup_{y: \rho(x-z-y) < \delta} \delta |\nabla F(y)| \leq 2^N \delta |\nabla F|_{N,1}^{**}(x) (1 + \rho(z))^N.$$

On the other hand,

$$\begin{aligned} (2.16) \quad \left(\int_{B(x-z, \delta)} |F(y)|^r dy \right)^{1/r} &\leq \left(\delta^{-\gamma} (\delta + \rho(z))^\gamma \int_{B(x, \delta + \rho(z))} |F(y)|^r dy \right)^{1/r} \\ &\leq \delta^{-\gamma/r} (\delta + \rho(z))^{\gamma/r} M(|F|^r)(x)^{1/r} \\ &\leq \delta^{-\gamma/r} (1 + \rho(z))^{\gamma/r} M(|F|^r)(x)^{1/r}. \end{aligned}$$

By (2.14), (2.15) and (2.16), we have

$$|F(x - z)| \leq C_r \delta^{-\gamma/r} (1 + \rho(z))^{\gamma/r} M(|F|^r)(x)^{1/r} + 2^N C_r \delta |\nabla F|_{N,1}^{**}(x) (1 + \rho(z))^N.$$

Thus, if $N = \gamma/r$, we see that

$$\frac{|F(x - z)|}{(1 + \rho(z))^N} \leq C_r \delta^{-N} M(|F|^r)(x)^{1/r} + 2^N C_r \delta |\nabla F|_{N,1}^{**}(x).$$

Taking the supremum in z over \mathbb{R}^n , we get the desired estimate. \square

Proof of Lemma 2.6. Let $(T_t f)(x) = f(A_t x)$. Then we note the following.

$$(T.1) \quad (T_t F_{N,R}^{**})(x) = (T_t F)_{N,tR}^{**}(x).$$

$$(T.2) \quad T_t(f * g)(x) = t^\gamma (T_t f) * (T_t g)(x).$$

$$(T.3) \quad T_t(M(f))(x) = M(T_t f)(x).$$

By (T.1) and (T.2) we have

$$T_t((f * \varphi_t)_{N,t-1}^{**})(x) = (T_t f * \varphi)_{N,1}^{**}(x).$$

Using Lemma 2.7, we see that

$$(2.17) \quad (T_t f * \varphi)_{N,1}^{**}(x) \leq C \delta^{-N} M(|T_t f * \varphi|^r)(x)^{1/r} + C \delta |T_t f * \nabla \varphi|_{N,1}^{**}(x).$$

Applying T_{t-1} to (2.17), we have

$$(2.18) \quad (f * \varphi_t)_{N,t-1}^{**}(x) \leq C \delta^{-N} T_{t-1}(M(|T_t f * \varphi|^r)(x)^{1/r}) + C \delta T_{t-1}(|T_t f * \nabla \varphi|_{N,1}^{**})(x).$$

From (T.2) and (T.3), it follows that

$$(2.19) \quad T_{t-1}(M(|T_t f * \varphi|^r)(x)^{1/r}) = M(|f * \varphi_t|^r)(x)^{1/r}.$$

Also, (T.1) and (T.2) imply

$$(2.20) \quad T_{t-1}(|T_t f * \nabla \varphi|_{N,1}^{**})(x) = |f * (\nabla \varphi)_t|_{N,t-1}^{**}(x).$$

Using (2.19) and (2.20) in (2.18), we get the conclusion of Lemma 2.6. \square

Proof of Lemma 2.5. Lemma 2.6 implies that

$$(2.21) \quad F(\varphi, f)(\cdot, t)_{N, t-1}^{**}(x) \leq C\delta^{-N} M(|f * \varphi_t|^r)(x)^{1/r} + C\delta |f * (\nabla \varphi)_t|_{N, t-1}^{**}(x),$$

where $r = \gamma/N$. Applying Lemma 2.3, we have

$$\begin{aligned} & |f * (\nabla \varphi)_t|_{N, t-1}^{**}(x) \\ & \leq C \sum_{j \geq 0} C_\varphi(\nabla \varphi, j, N) b^{-jN} F(\varphi, f)(\cdot, b^j t)_{N, (b^j t)-1}^{**}(x) + C D_\varphi(N) F(\varphi, f)(\cdot, t)_{N, t-1}^{**}(x). \end{aligned}$$

Thus by (2.21) and Hölder's inequality when $q > 1$ we have

$$\begin{aligned} (2.22) \quad F(\varphi, f)(\cdot, t)_{N, t-1}^{**}(x)^q & \leq C\delta^{-Nq} M(|f * \varphi_t|^r)(x)^{q/r} \\ & \quad + C\delta^q \sum_{j \geq 0} C_\varphi(\nabla \varphi, j, N)^q b^{-jNq} b^{-\tau c_q j} F(\varphi, f)(\cdot, b^j t)_{N, (b^j t)-1}^{**}(x)^q \\ & \quad + C\delta^q D_\varphi(N)^q F(\varphi, f)(\cdot, t)_{N, t-1}^{**}(x)^q. \end{aligned}$$

where $\tau > 0$, $c_q = 1$ when $q > 1$ and $c_q = 0$ when $0 < q \leq 1$. Integrating both sides of the inequality (2.22) over $(0, \infty)$ with respect to the measure dt/t and applying termwise integration on the right hand side, we see that

$$\begin{aligned} (2.23) \quad & \int_0^\infty F(\varphi, f)(\cdot, t)_{N, t-1}^{**}(x)^q \frac{dt}{t} \leq C\delta^{-Nq} \int_0^\infty M(|f * \varphi_t|^r)(x)^{q/r}(x) \frac{dt}{t} \\ & + C\delta^q \left[\sum_{j \geq 0} C_\varphi(\nabla \varphi, j, N)^q b^{-jNq} b^{-\tau c_q j} + D_\varphi(N)^q \right] \int_0^\infty F(\varphi, f)(\cdot, t)_{N, t-1}^{**}(x)^q \frac{dt}{t}. \end{aligned}$$

By (2.11) the sum in j on the right hand side of (2.23) is finite. By (2.2) and (2.3) with $\alpha = 0$ we easily see that the last integral on the right hand side of (2.23) is finite with $f \in \mathcal{S}(\mathbb{R}^n)$. Thus, along with (2.12) we see that the second term on the right hand side of (2.23) is finite. So, choosing δ sufficiently small, we get the conclusion. \square

We need the following version of the vector valued inequality for the Hardy-Littlewood maximal operator of Fefferman-Stein [7] with non-isotropic dilations and weights. The proof is essentially similar to the one in [7]. The case $P = E$ is stated in [16] (see [14] for related results).

Lemma 2.8. *Let $1 < \mu, \nu < \infty$ and $w \in A_\nu$. Then we have*

$$\left\| \left(\int_0^\infty M(F^t)(x)^\mu \frac{dt}{t} \right)^{1/\mu} \right\|_{\nu, w} \leq C \left(\int_{\mathbb{R}^n} \left(\int_0^\infty |F(x, t)|^\mu \frac{dt}{t} \right)^{\nu/\mu} w(x) dx \right)^{1/\nu}$$

for appropriate functions $F(x, t)$ on $\mathbb{R}^n \times (0, \infty)$ with $F^t(x) = F(x, t)$.

Proof of Theorem 2.4. By a change of variables we may assume that $\hat{\psi} = 0$ on $\{|\xi| \leq 1\} = \{\rho^*(\xi) \leq 1\}$. Define

$$\hat{\zeta}(\xi) = 1 - \sum_{j: b^j \leq r_2} \hat{\varphi}(A_{b^j}^* \xi) \hat{\eta}(A_{b^j}^* \xi).$$

Then $\text{supp}(\hat{\zeta}) \subset \{\rho^*(\xi) \leq 1\}$, $\hat{\zeta} = 1$ in $\{\rho^*(\xi) < r_1/r_2\}$. Since $\hat{\psi} = 0$ on $\{|\xi| \leq 1\}$, we have

$$\begin{aligned}\hat{\psi}(\xi) &= \sum_{j:bj \leq r_2} \hat{\psi}(\xi) \hat{\varphi}(A_{bj}^* \xi) \hat{\eta}(A_{bj}^* \xi) \\ &= \sum_{j:bj \leq r_2} \hat{\varphi}(A_{bj}^* \xi) \mathcal{F}(\alpha^{(bj)})(A_{bj}^* \xi),\end{aligned}$$

where $\alpha^{(bj)}(x) = (\psi)_{b^{-j}} * \eta(x)$.

Thus by an easier version of arguments for the proof of Lemma 2.3 we see that

$$|F(\psi, f)(x, t)| \leq C \sum_{j:bj \leq r_2} C(\psi, j, N) F(\varphi, f)(\cdot, b^j t)_{N, (bjt)^{-1}}^{**}(x),$$

from which it follows that

$$|F(\psi, f)(x, t)|^q \leq C \sum_{j:bj \leq r_2} C(\psi, j, N)^q b^{-\tau c_q j} F(\varphi, f)(\cdot, b^j t)_{N, (bjt)^{-1}}^{**}(x)^q,$$

where $\tau > 0$ and c_q is as in (2.22). Integrating with the measure dt/t over $(0, \infty)$, we have

$$\begin{aligned}(2.24) \quad & \int_0^\infty |F(\psi, f)(x, t)|^q \frac{dt}{t} \\ & \leq C \left[\sum_{j:bj \leq r_2} C(\psi, j, N)^q b^{-\tau c_q j} \right] \int_0^\infty F(\varphi, f)(\cdot, t)_{N, t^{-1}}^{**}(x)^q \frac{dt}{t}.\end{aligned}$$

By (2.13) the series on the right hand side of (2.24) converges. Let $0 < p, q < \infty$ and $w \in A_\infty$. If N is sufficiently large so that $r = \gamma/N < q, p$ and $w \in A_{pN/\gamma}$, from (2.24) and Lemma 2.5 it follows that

$$\begin{aligned}(2.25) \quad & \left\| \left(\int_0^\infty |f * \psi_t|^q \frac{dt}{t} \right)^{1/q} \right\|_{p,w} \leq C \left\| \left(\int_0^\infty M(|f * \varphi_t|^r)(x)^{q/r} \frac{dt}{t} \right)^{1/q} \right\|_{p,w} \\ & = C \left\| \left(\int_0^\infty M(|f * \varphi_t|^r)(x)^{q/r} \frac{dt}{t} \right)^{r/q} \right\|_{p/r,w}^{1/r} \\ & \leq C \left\| \left(\int_0^\infty |f * \varphi_t(x)|^q \frac{dt}{t} \right)^{1/q} \right\|_{p,w},\end{aligned}$$

where the last inequality follows from Lemma 2.8. This completes the proof of Theorem 2.4. \square

To conclude this section, we give a proof of the following results used above.

Lemma 2.9. *Let $\varphi \in B$. Suppose that $\psi \in L^1$, $\hat{\psi} \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and we have the estimates (2.3) with $\hat{\psi}$ in place of $\hat{\varphi}$ for all multi-indices α and all $\tau > 0$. Let $L, J \geq 0$. Then*

- (1) $\sup_{j:bj \leq J} C_\varphi(\psi, j, L) b^{-j\tau} < \infty$ for any $\tau > 0$, where $C_\varphi(\psi, j, L) = C(\psi, j, L)$ is as in (2.8) with ψ in place of $\partial_k \varphi$.
- (2) $D_\varphi(\Xi_k, L) < \infty$, $1 \leq k \leq n$, where $\Xi_k(\xi) = 2\pi i \xi_k$ as above and $D_\varphi(\Xi_k, L) = D(\Xi_k, L)$ is as in (2.9).

Proof. To prove part (1), since $1 + \rho(x) \leq c(1 + |x|)$ by (P.4), we have

$$(1 + |x|)^{[n/2]+1} C_0(\psi, t, L, x) \leq C \left| \int \hat{\psi}(A_{t^{-1}}^* \xi) \hat{\eta}(\xi) e^{2\pi i \langle x, \xi \rangle} d\xi \right| + C \sup_{|\alpha|=L+[n/2]+1} \left| \int \partial_\xi^\alpha \left[\hat{\psi}(A_{t^{-1}}^* \xi) \hat{\eta}(\xi) \right] e^{2\pi i \langle x, \xi \rangle} d\xi \right|,$$

where $C_0(\psi, t, L, x)$ is as in (2.6) with ψ in place of $\partial_k \varphi$ and $[a]$ denotes the largest integer not exceeding a . We recall that $\hat{\eta} \in C^\infty(\mathbb{R}^n)$ with support in $\{r_1 < \rho^*(\xi) < r_2\}$, which is in Lemma 2.1. It is known that $\|A_{t^{-1}}^*\| \leq t^{-\kappa}$ for $t \in (0, 1]$ with some $\kappa \geq 1$. Thus (2.3) for $\hat{\psi}$ implies

$$\left| \partial_\xi^\alpha \left[\hat{\psi}(A_{t^{-1}}^* \xi) \hat{\eta}(\xi) \right] \right| \leq C_{\alpha, M} t^\tau, \quad 0 < t \leq M,$$

for any $M > 0$, if $|\alpha| = L + [n/2] + 1$ or $\alpha = 0$. Thus

$$C_0(\psi, t, L, x) \leq C(1 + |x|)^{-[n/2]-1} G(x)$$

with some $G \in L^2$ such that $\|G\|_2 \leq Ct^\tau$, and hence, the Schwarz inequality implies

$$(2.26) \quad \int_{\mathbb{R}^n} C_0(\psi, t, L, x) dx \leq Ct^\tau,$$

since $[n/2] + 1 > n/2$. From (2.26) with $t = b^j$ we obtain the conclusion of part (1).

Similarly, we can see that

$$\int_{\mathbb{R}^n} D(\Xi_k, L, x) dx < \infty,$$

where $D(\Xi_k, L, x)$ is as in (2.7). This completes the proof of part (2). \square

3. LITTLEWOOD-PALEY FUNCTIONS AND PARABOLIC HARDY SPACES

As an application of Theorem 2.4 we have the following.

Corollary 3.1. *Let $0 < p \leq 1$. Suppose that $\varphi \in B$. Then we have*

$$\|f\|_{H^p} \leq C_p \|g_\varphi(f)\|_p$$

for $f \in H^p(\mathbb{R}^n) \cap \mathcal{S}(\mathbb{R}^n)$ with a positive constant C_p independent of f .

Let \mathcal{H} be the Hilbert space of functions $u(t)$ on \mathbb{R}_+ such that $\|u\|_{\mathcal{H}} = \left(\int_0^\infty |u(t)|^2 dt/t \right)^{1/2} < \infty$. Let $L_{\mathcal{H}}^q(\mathbb{R}^n)$ be the Lebesgue space of functions $h(y, t)$ with the norm

$$\|h\|_{q, \mathcal{H}} = \left(\int_{\mathbb{R}^n} \|h^y\|_{\mathcal{H}}^q dy \right)^{1/q},$$

where $h^y(t) = h(y, t)$.

Let $0 < p \leq 1$. We consider the parabolic Hardy space of functions on \mathbb{R}^n with values in \mathcal{H} , which is denoted by $H_{\mathcal{H}}^p(\mathbb{R}^n)$. Choose $\Phi \in \mathcal{S}(\mathbb{R}^n)$ as in the definition of H^p in Section 1. Let $h \in L_{\mathcal{H}}^2(\mathbb{R}^n)$. We say $h \in H_{\mathcal{H}}^p(\mathbb{R}^n)$ if $\|h\|_{H_{\mathcal{H}}^p} = \|h^*\|_{L^p} < \infty$ with

$$h^*(x) = \sup_{s>0} \left(\int_0^\infty |\Phi_s * h^t(x)|^2 \frac{dt}{t} \right)^{1/2},$$

where we write $h^t(x) = h(x, t)$.

To prove Corollary 3.1 we need the following.

Lemma 3.2. *Let $\hat{\psi}$ be a function of $\mathcal{S}(\mathbb{R}^n)$ with support in $\{1 \leq \rho^*(\xi) \leq 2\}$. Suppose that*

$$\int_0^\infty |\hat{\psi}(A_t^* \xi)|^2 \frac{dt}{t} = 1 \quad \text{for all } \xi \neq 0.$$

*Let $F(y, t) = f * \psi_t(y)$ with $f \in H^p(\mathbb{R}^n) \cap \mathcal{S}(\mathbb{R}^n)$, $0 < p \leq 1$. Then $F \in H_{\mathcal{H}}^p(\mathbb{R}^n)$ and*

$$\|f\|_{H^p} \leq C \|F\|_{H_{\mathcal{H}}^p}.$$

Define

$$E_\psi^\epsilon(h)(x) = \int_0^\infty \int_{\mathbb{R}^n} \psi_t(x-y) h_{(\epsilon)}(y, t) dy \frac{dt}{t},$$

where $h \in L_{\mathcal{H}}^2(\mathbb{R}^n)$ and $h_{(\epsilon)}(y, t) = h(y, t) \chi_{(\epsilon, \epsilon^{-1})}(t)$, $0 < \epsilon < 1$, and we assume that $\psi \in L^1(\mathbb{R}^n)$ with the cancellation (1.2). Here χ_S denotes the characteristic function of a set S .

We apply the following result in proving Lemma 3.2.

Lemma 3.3. *Suppose that $\psi \in \mathcal{S}(\mathbb{R}^n)$ and $\text{supp } \hat{\psi} \subset \{1 \leq \rho^*(\xi) \leq 2\}$. Then*

$$\sup_{\epsilon \in (0,1)} \|E_\psi^\epsilon(h)\|_{H^p} \leq C \|h\|_{H_{\mathcal{H}}^p}, \quad 0 < p \leq 1.$$

Let a be a (p, ∞) atom in $H_{\mathcal{H}}^p(\mathbb{R}^n)$:

- (i) $(\int_0^\infty |a(x, t)|^2 dt/t)^{1/2} \leq |B|^{-1/p}$, where B is a ball in \mathbb{R}^n with respect to ρ ;
- (ii) $\text{supp}(a(\cdot, t)) \subset B$ for all $t > 0$, where B is as in (i);
- (iii) $\int_{\mathbb{R}^n} a(x, t) x^\alpha dx = 0$ for all $t > 0$ and α such that $|\alpha| \leq [\gamma(1/p - 1)]$, where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index with $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$.

To prove Lemma 3.3 we use the following atomic decomposition.

Lemma 3.4. *Let $h \in L_{\mathcal{H}}^2(\mathbb{R}^n)$. Suppose that $h \in H_{\mathcal{H}}^p(\mathbb{R}^n)$. Then we can find a sequence $\{a_k\}$ of (p, ∞) atoms in $H_{\mathcal{H}}^p(\mathbb{R}^n)$ and a sequence $\{\lambda_k\}$ of positive numbers such that $h = \sum_{k=1}^\infty \lambda_k a_k$ in $H_{\mathcal{H}}^p(\mathbb{R}^n)$ and in $L_{\mathcal{H}}^2(\mathbb{R}^n)$ and such that $\sum_{k=1}^\infty \lambda_k^p \leq C \|h\|_{H_{\mathcal{H}}^p}^p$ with a constant C independent of h .*

We can find in [12] a proof of the atomic decomposition for $H^p(\mathbb{R}^n)$ (see also [3], [10] and [18]). The vector valued case can be treated similarly.

Proof of Lemma 3.3. Let a be a (p, ∞) atom in $H_{\mathcal{H}}^p(\mathbb{R}^n)$ supported on the ball B of the definition of the atom. We choose a non-negative C^∞ function Φ on \mathbb{R}^n supported on $\{|x| < 1\}$ with $\int \Phi(x) dx = 1$. We prove

$$(3.1) \quad \int_{\mathbb{R}^n} \sup_{s>0} |\Phi_s * E_\psi^\epsilon(a)(x)|^p dx \leq C.$$

To show this, by applying translation and dilation arguments, we may assume that $B = B(0, 1)$. Let $\tilde{B} = B(0, 2)$. Then $2\rho(y) \leq \rho(x)$ if $y \in B$ and $x \in \mathbb{R}^n \setminus \tilde{B}$. Let $\Psi_{s,t} = \Phi_s * \psi_t$, $s, t > 0$. We note that $\Phi_s * \psi_t = (\Phi_{s/t} * \psi)_t$ and $\Phi_u * \psi$, $u > 0$, belongs to a bounded subset of the topological vector space $\mathcal{S}(\mathbb{R}^n)$, which can easily be seen by noting that $\mathcal{F}(\Phi_u * \psi)(\xi) = \hat{\Phi}(A_u^* \xi) \hat{\psi}(\xi)$ and recalling that $\hat{\psi}(\xi)$ is supported on $\{1 \leq \rho^*(\xi) \leq 2\}$.

Let $P_x(y)$ be the Taylor polynomial in y of order $M = [\gamma(1/p - 1)]$ at 0 for $\Phi_{s/t} * \psi(x - y)$. Then, if $\rho(x) \geq 2\rho(y)$ and we have

$$|\Phi_{s/t} * \psi(x - y) - P_x(y)| \leq C |y|^{M+1} (1 + \rho(x))^{-L},$$

where L is sufficiently large, which will be specified below, and the constant C is independent of s, t, x, y . This implies

$$|\Psi_{s,t}(x-y) - t^{-\gamma} P_{A_{t^{-1}}x}(A_{t^{-1}}y)| \leq C t^{-\gamma} |A_{t^{-1}}y|^{M+1} (1 + \rho(x)/t)^{-L}.$$

Thus, using the properties of an atom and the Schwarz inequality, for $x \in \mathbb{R}^n \setminus \tilde{B}$ we see that

(3.2)

$$\begin{aligned} |\Phi_s * E_\psi^\epsilon(a)(x)| &= \left| \iint (\Psi_{s,t}(x-y) - t^{-\gamma} P_{A_{t^{-1}}x}(A_{t^{-1}}y)) a_{(\epsilon)}(y, t) dy \frac{dt}{t} \right| \\ &\leq \int_B \left(\int_0^\infty |\Psi_{s,t}(x-y) - t^{-\gamma} P_{A_{t^{-1}}x}(A_{t^{-1}}y)|^2 \frac{dt}{t} \right)^{1/2} \left(\int_0^\infty |a(y, t)|^2 \frac{dt}{t} \right)^{1/2} dy \\ &\leq C |B|^{-1/p} \int_B \left(\int_0^\infty (t^{-\gamma} |A_{t^{-1}}y|^{M+1} (1 + \rho(x)/t)^{-L})^2 \frac{dt}{t} \right)^{1/2} dy. \end{aligned}$$

Now we show that

$$(3.3) \quad I(x, y) := \int_0^\infty (t^{-\gamma} |A_{t^{-1}}y|^{M+1} (1 + \rho(x)/t)^{-L})^2 \frac{dt}{t} \leq C \rho(x)^{-2(\gamma+M+1)}$$

for $y \in B$, $x \in \mathbb{R}^n \setminus \tilde{B}$, if L is sufficiently large. We first see that

$$(3.4) \quad I(x, y) = \rho(x)^{-2\gamma} \int_0^\infty (t^{-\gamma} |A_{(\rho(x)t)^{-1}}y|^{M+1} (1 + t^{-1})^{-L})^2 \frac{dt}{t}.$$

By (P.6) we have

$$\begin{aligned} (3.5) \quad &\int_{\rho(x)^{-1}}^\infty (t^{-\gamma} |A_{(\rho(x)t)^{-1}}y|^{M+1} (1 + t^{-1})^{-L})^2 \frac{dt}{t} \\ &\leq C |y|^{2(M+1)} \rho(x)^{-2(M+1)} \int_{\rho(x)^{-1}}^\infty t^{-2(\gamma+M+1)} (1 + t^{-1})^{-2L} \frac{dt}{t} \\ &\leq C |y|^{2(M+1)} \rho(x)^{-2(M+1)} \end{aligned}$$

if $L > \gamma + M + 1$. If $s \geq 1$, $|A_s y| \leq C s^\kappa |y|$ for some $\kappa \geq 1$. Thus

$$\begin{aligned} (3.6) \quad &\int_0^{\rho(x)^{-1}} (t^{-\gamma} |A_{(\rho(x)t)^{-1}}y|^{M+1} (1 + t^{-1})^{-L})^2 \frac{dt}{t} \\ &\leq C |y|^{2(M+1)} \rho(x)^{-2\kappa(M+1)} \int_0^{\rho(x)^{-1}} t^{-2(\gamma+\kappa(M+1))} (1 + t^{-1})^{-2L} \frac{dt}{t} \\ &\leq C |y|^{2(M+1)} \rho(x)^{-2\kappa(M+1)}, \end{aligned}$$

if $L > \gamma + \kappa(M + 1)$. By (3.4), (3.5) and (3.6) we obtain (3.3). From (3.2) and (3.3) we have

$$|\Phi_s * E_\psi^\epsilon(a)(x)| \leq C \rho(x)^{-(\gamma+M+1)}$$

for $x \in \mathbb{R}^n \setminus \tilde{B}$.

Since $p > \gamma/(\gamma + M + 1)$, it follows that

$$(3.7) \quad \int_{\mathbb{R}^n \setminus \tilde{B}} \sup_{s>0} |\Phi_s * E_\psi^\epsilon(a)(x)|^p dx \leq C \int_{\mathbb{R}^n \setminus \tilde{B}} \rho(x)^{-p(\gamma+M+1)} dx \leq C$$

(see [6]).

Using $\int_0^\infty |\hat{\psi}(A_t^* \xi)|^2 dt/t \leq C$, by duality we can easily see that

$$\sup_{\epsilon \in (0,1)} \|E_\psi^\epsilon(h)\|_2 \leq C \|h\|_{L_{\mathcal{H}}^2}, \quad h \in L_{\mathcal{H}}^2(\mathbb{R}^n).$$

So, by Hölder's inequality and the properties (i), (ii) of a , we get

$$\begin{aligned} (3.8) \quad \int_{\tilde{B}} \sup_{s>0} |\Phi_s * E_\psi^\epsilon(a)(x)|^p dx &\leq C \left(\int_{\tilde{B}} |M(E_\psi^\epsilon(a))(x)|^2 dx \right)^{p/2} \\ &\leq C \left(\int_B \int_0^\infty |a(y,t)|^2 \frac{dt}{t} dy \right)^{p/2} \\ &\leq C. \end{aligned}$$

Combining (3.7) and (3.8), we have (3.1). By Lemma 3.4 and (3.1) we can prove

$$\int_{\mathbb{R}^n} \sup_{s>0} |\Phi_s * E_\psi^\epsilon(h)(x)|^p dx \leq C \|h\|_{H_{\mathcal{H}}^p}^p.$$

This completes the proof. \square

Proof of Lemma 3.2. By using the atomic decomposition for $H^p(\mathbb{R}^n)$, we can prove the fact that $F \in H_{\mathcal{H}}^p(\mathbb{R}^n)$ similarly to the proof of Lemma 3.3 (see [19, Lemma 3.6]).

We note that

$$E_{\tilde{\psi}}^\epsilon(F)(x) = \int_\epsilon^{\epsilon^{-1}} \int_{\mathbb{R}^n} \psi_t * f(y) \bar{\psi}_t(y-x) dy \frac{dt}{t} = \int_{\mathbb{R}^n} \Psi^{(\epsilon)}(x-z) f(z) dz,$$

where

$$\Psi^{(\epsilon)}(x) = \int_\epsilon^{\epsilon^{-1}} \int_{\mathbb{R}^n} \psi_t(x+y) \bar{\psi}_t(y) dy \frac{dt}{t}.$$

We have

$$\widehat{\Psi^{(\epsilon)}}(\xi) = \int_\epsilon^{\epsilon^{-1}} \hat{\psi}(A_t^* \xi) \widehat{\tilde{\psi}}(-A_t^* \xi) \frac{dt}{t} = \int_\epsilon^{\epsilon^{-1}} |\hat{\psi}(A_t^* \xi)|^2 \frac{dt}{t}.$$

This and Lemma 3.3 imply

$$\|f\|_{H^p} \leq C \liminf_{\epsilon \rightarrow 0} \|E_{\tilde{\psi}}^\epsilon(F)\|_{H^p} \leq C \|F\|_{H_{\mathcal{H}}^p}.$$

\square

We also need the following result to prove Corollary 3.1.

Lemma 3.5. *Let $\eta \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\text{supp}(\hat{\eta}) \subset \{1/2 \leq \rho^*(\xi) \leq 4\}$ and $\hat{\eta}(\xi) = 1$ on $\{1 \leq \rho^*(\xi) \leq 2\}$. Let ψ be as in Lemma 3.2. Suppose that $\Phi \in \mathcal{S}(\mathbb{R}^n)$ satisfies $\int_{\mathbb{R}^n} \Phi(x) dx = 1$ and $\text{supp}(\Phi) \subset B(0, 1)$. Then for $p, q > 0$ and $f \in \mathcal{S}(\mathbb{R}^n)$ we have*

$$\left\| \left(\int_0^\infty \sup_{s>0} |\Phi_s * \psi_t * f|^q \frac{dt}{t} \right)^{1/q} \right\|_p \leq C \left\| \left(\int_0^\infty |\eta_t * f|^q \frac{dt}{t} \right)^{1/q} \right\|_p.$$

Proof. Since $\hat{\Phi}(A_s^* \xi) \hat{\psi}(A_t^* \xi) = \hat{\Phi}(A_s^* \xi) \hat{\psi}(A_t^* \xi) \hat{\eta}(A_t^* \xi)$, we have

$$\begin{aligned} |\Phi_s * \psi_t * f(x)| &\leq (f * \eta_t)_{N,t^{-1}}^{**}(x) \int_{\mathbb{R}^n} |\Phi_s * \psi_t(w)| (1 + t^{-1} \rho(w))^N dw \\ &= (f * \eta_t)_{N,t^{-1}}^{**}(x) \int_{\mathbb{R}^n} |\Phi_{s/t} * \psi(w)| (1 + \rho(w))^N dw \\ &\leq C_N (f * \eta_t)_{N,t^{-1}}^{**}(x) \end{aligned}$$

for any $N > 0$, where C_N is independent of s, t . This follows from the observation that $\Phi_{s/t} * \psi$, $s, t > 0$, belongs to a bounded subset of the topological vector space $\mathcal{S}(\mathbb{R}^n)$, as in the proof of Lemma 3.3. Thus

$$(3.9) \quad \left(\int_0^\infty \sup_{s>0} |\Phi_s * \psi_t * f(x)|^q \frac{dt}{t} \right)^{1/q} \leq C \left(\int_0^\infty |(f * \eta_t)_{N,t^{-1}}^{**}(x)|^q \frac{dt}{t} \right)^{1/q}.$$

By (3.9) and Lemma 2.5 with η in place of φ , we have

$$\left(\int_0^\infty \sup_{s>0} |\Phi_s * \psi_t * f(x)|^q \frac{dt}{t} \right)^{1/q} \leq C \left(\int_0^\infty M(|f * \eta_t|^r)(x) (x)^{q/r} \frac{dt}{t} \right)^{1/q},$$

where $N = \gamma/r$. This and Lemma 2.8 prove Lemma 3.5 as in (2.25). \square

Proof of Corollary 3.1. Let η be as in Lemma 3.5. Applying Lemma 3.2 and Lemma 3.5 with $q = 2$ and $p \in (0, 1]$, we see that

$$\|f\|_{H^p} \leq C \|g_\eta(f)\|_p, \quad f \in H^p(\mathbb{R}^n) \cap \mathcal{S}(\mathbb{R}^n),$$

which combined with Theorem 2.4 with $q = 2$, $p \in (0, 1]$, $w = 1$ and with η in place of ψ proves Corollary 3.1. \square

Proof of Theorem 1.1. Let φ be as in Theorem 1.1. Let $0 < p \leq 1$. The inequality

$$\|g_\varphi(f)\|_p \leq C \|f\|_{H^p}$$

can be proved similarly to the proof of the statement $F \in H_{\mathcal{H}}^p(\mathbb{R}^n)$ in Lemma 3.2 for $f \in H^p(\mathbb{R}^n) \cap \mathcal{S}(\mathbb{R}^n)$ by using the atomic decomposition for $H^p(\mathbb{R}^n)$. This and Corollary 3.1 imply

$$c_1 \|f\|_{H^p} \leq \|g_\varphi(f)\|_p \leq c_2 \|f\|_{H^p}$$

for $f \in H^p(\mathbb{R}^n) \cap \mathcal{S}(\mathbb{R}^n)$, from which the conclusion of Theorem 1.1 follows by arguments similar to the one in [19, pp. 149–150], since $H^p(\mathbb{R}^n) \cap \mathcal{S}(\mathbb{R}^n)$ is dense in $H^p(\mathbb{R}^n)$ (see [5]). \square

It is not difficult to see that we have discrete parameter versions of Theorems 1.1 and 2.4. To conclude this note we remark the following results.

Theorem 3.6. *Let φ be as in Theorem 1.1 and $0 < p \leq 1$. Then, there exist positive constants c_1, c_2 such that*

$$c_1 \|f\|_{H^p} \leq \left\| \left(\sum_{j=-\infty}^{\infty} |f * \varphi_{bj}|^2 \right)^{1/2} \right\|_p \leq c_2 \|f\|_{H^p}$$

for $f \in H^p(\mathbb{R}^n)$.

Theorem 3.7. *Let $0 < p, q < \infty$ and $w \in A_\infty$. Suppose that φ and ψ fulfill the hypotheses of Theorem 2.4. Then we have, for $f \in \mathcal{S}(\mathbb{R}^n)$,*

$$\left\| \left(\sum_{j=-\infty}^{\infty} |f * \psi_{bj}|^q \right)^{1/q} \right\|_{p,w} \leq C \left\| \left(\sum_{j=-\infty}^{\infty} |f * \varphi_{bj}|^q \right)^{1/q} \right\|_{p,w}.$$

REFERENCES

- [1] A. Benedek, A. P. Calderón and R. Panzone, *Convolution operators on Banach space valued functions*, Proc. Nat. Acad. Sci. U. S. A. **48** (1962), 356–365.
- [2] A. P. Calderón, *Inequalities for the maximal function relative to a metric*, Studia Math. **57** (1976), 297–306.
- [3] A. P. Calderón, *An atomic decomposition of distributions in parabolic H^p spaces*, Advances in Math. **25** (1977), 216–225.
- [4] A. P. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution*, Advances in Math. **16** (1975), 1–64.
- [5] A. P. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution. II*, Advances in Math. **24** (1977), 101–171.
- [6] O. N. Capri, *On an inequality in the theory of parabolic H^p spaces*, Revista de la Unión Matemática Argentina **32** (1985), 17–28.
- [7] C. Fefferman and E. M. Stein, *Some maximal inequalities*, Amer. J. Math. **93** (1971), 107–115.
- [8] C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), 137–193.
- [9] G. B. Folland and E. M. Stein, *Hardy Spaces on Homogeneous Groups*, Princeton Univ. Press, Princeton, N.J. 1982.
- [10] J. Garcia-Cuerva and J.L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland, Amsterdam, New York, Oxford, 1985.
- [11] L. Grafakos, *Modern Fourier Analysis*, Third Edition, Graduate Texts in Math., no 250, Springer, New York, 2014.
- [12] R. H. Latter and A. Uchiyama, *The atomic decomposition for parabolic H^p spaces*, Trans. Amer. Math. Soc. **253** (1979), 391–398.
- [13] J. Peetre, *On spaces of Triebel-Lizorkin type*, Ark. Mat. **13** (1975), 123–130.
- [14] J. L. Rubio de Francia, F. J. Ruiz and J. L. Torrea, *Calderón-Zygmund theory for operator-valued kernels*, Adv. in Math. **62** (1986), 7–48.
- [15] S. Sato, *Remarks on square functions in the Littlewood-Paley theory*, Bull. Austral. Math. Soc. **58** (1998), 199–211.
- [16] S. Sato, *Vector valued inequalities and Littlewood-Paley operators on Hardy spaces*, preprint (2015).
- [17] E. M. Stein and G. Weiss, *Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, 1971.
- [18] J. -O. Strömberg and A. Torchinsky, *Weighted Hardy spaces*, Lecture Notes in Math. 1381, Springer-Verlag, Berlin Heidelberg New York London Paris Tokyo Hong Kong, 1989.
- [19] A. Uchiyama, *Characterization of $H^p(\mathbb{R}^n)$ in terms of generalized Littlewood-Paley g -functions*, Studia Math. **81** (1985), 135–158.

DEPARTMENT OF MATHEMATICS, FACULTY OF EDUCATION, KANAZAWA UNIVERSITY, KANAZAWA 920-1192, JAPAN

E-mail address: shuichi@kenroku.kanazawa-u.ac.jp